AIAA JOURNAL
Vol. 46, No. 10, October 2008

Frequency Response of a Combined Structure Using a Modified Finite Element Method

Philip D. Cha*
Harvey Mudd College, Claremont, California 91711
and
Xiang Zhou†
Tsinghua University, 100084 Beijing, People's Republic of China

DOI: 10.2514/1.25845

A modified finite element method is proposed to analyze the frequency response of an arbitrarily supported linear structure carrying various lumped attachments. This is accomplished by updating the finite element mass and stiffness matrices of the linear structure without any lumped attachments using its exact natural frequencies and mode shapes, so that the eigensolutions of the updated linear structure coincide with the exact eigendata. Once the updated matrices for the linear structure are found, the finite element assembling technique is then exploited to include the lumped attachments by adding their parameters to the appropriate elements in the modified mass and stiffness matrices. After assembling the global mass and stiffness matrices, the frequency response of the combined system can be easily obtained. Numerical experiments show that, using only a few finite elements, the proposed scheme returns natural frequencies and frequency responses that are nearly identical to those obtained by using a finite element model with a very fine mesh. The new method is easy to apply and efficient to use, and it can be extended to determine the frequency response of any combined linear structure over any specified range of excitation frequencies.

Nomenclature

E	=	Young's modulus of a beam
$\mathbf{F}$	=	vector of input amplitudes
I	=	area moment of inertia of the cross section of a beam
$[I]$	=	identify matrix
J_K	=	objective function used to update the stiffness matrix of the linear structure
J_M	=	objective function used to update the mass matrix of the linear structure
j	=	imaginary unit
$[K]$	=	updated finite element stiffness matrix of the linear structure
$[K_0]$	=	initial finite element stiffness matrix of the linear structure
$[\mathcal{K}]$	=	global stiffness matrix of the combined system
k_i	=	ith translational spring constant
k_t	=	torsional spring constant
L	=	length of a beam
$[M]$	=	updated finite element mass matrix of the linear structure
$[M_0]$	=	initial finite element mass matrix of the linear structure
$[\mathcal{M}]$	=	global mass matrix of the combined system
$[m]$	=	normalized mass matrix
m_i	=	ith lumped mass
N	=	number of generalized coordinates
n	=	number of finite elements
$\mathbf{p}_i$	=	ith eigenvector of the combined system
$\mathbf{q}$	=	vector of generalized coordinates of the combined system
$\bar{\mathbf{q}}$	=	vector of amplitudes of the generalized coordinates
$[U]$	=	exact modal matrix of the linear structure
$v_i(x)$	=	ith exact eigenfunction of the linear structure evaluated at x
x_a	=	attachment location
x_f	=	forcing location
x_{ai}	=	ith attachment location
γ	=	structural damping coefficient
$[\Delta]$	=	modification to the stiffness matrix
ϵ_i	=	relative error parameter for the ith natural frequency of the combined system
$\theta_i(x)$	=	slope of the ith exact eigenfunction evaluated at x
$[\Lambda]$	=	diagonal matrix consisting of the exact eigenvalues of the linear structure
λ_i	=	ith exact eigenvalue of the linear structure
λ_{ij}	=	Lagrange multipliers used to enforce the orthogonality conditions of the eigenvectors with respect to $[M]$
λ_{Kij}	=	Lagrange multipliers used to enforce the generalized eigenvalue problem
λ_{0ij}	=	Lagrange multipliers used to enforce the orthogonality conditions of the eigenvectors with respect to $[K]$
λ_{Sij}	=	Lagrange multipliers used to enforce the symmetry condition for $[K]$
ρ	=	mass per unit length of a beam
ω	=	excitation frequency
ω_i	=	ith natural frequency of the combined system
$(\omega_i)_{100}$	=	ith natural frequency of the combined system obtained using finite element method whereby the beam is discretized into 100 equal elements
ω_i'	=	ith exact natural frequency of the linear structure

Received 14 June 2006; revision received 30 April 2008; accepted for publication 16 June 2008.

*Professor, Department of Engineering.
†Graduate Student, School of Aerospace; currently Doctorate Program, Oxford University.

Introduction

RESEARCHERS have performed frequency analyses on combined dynamic systems consisting of a linear structure carrying any number of lumped attachments extensively for years, and hence only a few selected recent references are given here [1–16]. Commonly used analytical approaches include the assumed-modes method [13,16], the Lagrange multipliers formalism

[3,10,12], the dynamic Green's function approach [4,9,11], the Laplace transform with respect to the spatial variable approach [2,15], and the analytical-and-numerical-combined method [6,14]. However, due to their complexity, these methods have not been as widely adopted as the finite element method (FEM).

In this paper, a modified finite element method is proposed that can be effectively used to obtain the frequency response of a combined system consisting of a linear structure carrying various lumped attachments. To obtain the frequency response of such a system using FEM at the higher excitation frequencies, one typically refines the mesh of the linear structure until the accuracy criteria are satisfied. Although conceptually straightforward, this approach of refining the mesh to determine frequency response at high excitation frequencies is costly and time consuming. The slow convergence can be attributed to the fact that many elements are often needed to model the linear structure itself so that the higher natural frequencies (which lie within the range of excitation frequencies of interest) of the discretized linear structure match well with the exact solutions.

To expedite convergence and to obtain sufficiently accurate results with the least cost, a new scheme is introduced to improve the finite element mass and stiffness matrices of the linear structure without any lumped attachments such that the eigendata of the updated finite element model of the linear structure coincide with the exact eigensolution. Once the system matrices of the linear structure have been updated, the finite element assembling technique is exploited to account for the lumped attachments, after which, the frequency response of the combined system can be easily determined. Numerical experiments show that, by applying the proposed discretization scheme, one can use a coarse mesh to obtain the frequency response of a combined system accurately over any specified range of excitation frequencies.

Theory

In the field of model updating, Berman and Nagy [17] developed a method to update the analytical mass and stiffness matrices of a structure using test data. The method returns a set of minimum changes in the system matrices such that the eigensolutions coincide with the test measurements. In this paper, the same approach is employed to determine the frequency response of a combined system consisting of a linear structure carrying lumped attachments. In particular, knowing the range of excitation frequencies of interest, a properly chosen set of the exact eigendata of the linear structure is first used to modify its finite element mass and stiffness matrices. Once the system matrices of the linear structure have been updated, one can easily include the lumped attachments by exploiting the finite element assembling technique, and then determine the frequency response of the combined system at any point along the linear structure over the desired range of excitation frequencies.

Figure 1 shows a combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments, including a grounded k_1 at x_{a1}, m_1 at x_{a2}, an oscillator of parameters m_2 and k_2 with a rigid body degree of freedom at x_{a3}, a grounded k_t at x_4, and an oscillator of parameters m_3 and k_3 with no rigid body degree of freedom at x_5. Assume the linear structure has been discretized into uniform finite elements and possesses N generalized coordinates, whose system matrices are $[M_0]$ and $[K_0]$ (both of size $N \times N$). The eigensolutions of the associated generalized eigenvalue problem correspond to the modes of vibration of the discretized linear structure. Suppose the exact eigendata of the linear structure are known. They can be exploited to improve or update the mass and stiffness matrices of the linear structure such that the modified system returns eigensolutions that are exact, even for finite N.

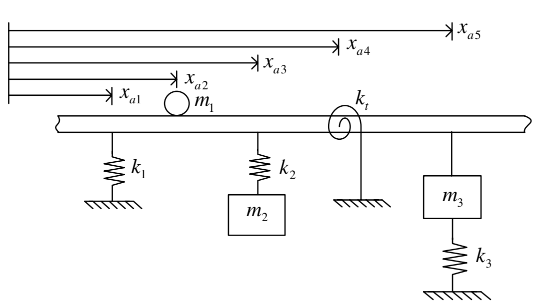

Fig. 1 Combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments.

To find $[M]$ of the linear structure, the following objective function is minimized:

$$J_M = \|[M_0]^{-1/2}([M]-[M_0])[M_0]^{-1/2}\| + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}([U]^T[M][U]-[I])_{ij} \tag{1}$$

where $\|[A]\|$ denotes the sum of the squares of all elements of matrix $[A]$, and elements of $[U]$ are obtained from the exact eigenfunctions of the linear structure. Equation (1) is differentiated with respect to the elements of $[M]$ and set to zero, and the undetermined Lagrange multipliers are obtained by enforcing the constraint equation $[U]^T[M][U] = [I]$. The minimization procedure results in the expression for the updated mass matrix as follows (see [18] for detailed derivation):

$$[M] = [M_0] + [M_0][U][m]^{-1}([I]-[m])[m]^{-1}[U]^T[M_0] \tag{2}$$

where

$$[m] = [U]^T[M_0][U] \tag{3}$$

Once $[M]$ has been computed using Eq. (2), $[K]$ can be obtained by minimizing yet another objective function of the form

$$\begin{aligned} J_K = {} & \|[M]^{-1/2}([K]-[K_0])[M]^{-1/2}\| \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{Kij}([K][U]-[M][U][\Lambda])_{ij} \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{0ij}([U]^T[K][U]-[\Lambda])_{ij} \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{Sij}([K]-[K]^T)_{ij} \end{aligned} \tag{4}$$

Equation (4) is differentiated with respect to the elements of $[K]$ and set to zero. Using the constraint equations $[K][U] = [M][U][\Lambda]$, $[U]^T[K][U] = [\Lambda]$, and $[K] = [K]^T$ to eliminate the undetermined Lagrange multipliers, one obtains the following expression for the updated stiffness matrix (see [19] for detailed derivation):

$$[K] = [K_0] + [\Delta] + [\Delta]^T \tag{5}$$

where

$$[\Delta] = \frac{1}{2}[M][U]([U]^T[K_0][U] + [\Lambda])[U]^T[M] - [K_0][U][U]^T[M] \tag{6}$$

Equations (2) and (5) lead to updated mass and stiffness matrices whose eigensolutions coincide with the exact eigendata of the linear structure. The proposed modification scheme returns an updated model without iteration, and requires only simple matrix multiplications. Once these updated matrices are obtained, the lumped attachments are added to the updated system matrices to form $[\mathcal{M}]$ and $[\mathcal{K}]$ using the traditional finite element assembling technique, and the ω_i of the combined assembly are obtained by solving the following generalized eigenvalue problem:

$$[\mathcal{K}]\mathbf{p}_i = \omega_i^2[\mathcal{M}]\mathbf{p}_i \tag{7}$$

When the combined system is subjected to external harmonic inputs with ω, its equations of motion have the form

$$[\mathcal{M}]\ddot{\mathbf{q}} + (1+j\gamma)[\mathcal{K}]\mathbf{q} = \mathbf{F}e^{j\omega t} \tag{8}$$

where $\gamma = 0.001$ is introduced into the combined system to guarantee finite response at all excitation frequencies. The frequency response of the combined system at any point along the linear structure is obtained by solving

$$\{(1 + j\gamma)[\mathcal{K}] - \omega^2[\mathcal{M}]\}\bar{\mathbf{q}} = \mathbf{F} \tag{9}$$

Results

The modified finite element method proposed in this paper is inspired by the work done in the field of model updating, where the objective is to improve the analytical mass and stiffness matrices of a structure based on measured normal modes and natural frequencies. Using the Lagrange multipliers formalism, Eqs. (2) and (5) were derived, whereby a set of minimum changes in the mass and stiffness matrices are identified such that the eigensolutions of the updated model match the measured normal modes and natural frequencies. In this paper, the exact eigenfunctions, instead of the measured modes of vibration, are used to update the mass and stiffness matrices of an FE model of a linear structure. In theory, Eqs. (2) and (5) can be used to find the minimum changes in the coarse FE model of any linear structure so that its eigensolutions match the exact modes of vibration. For simplicity, the linear structure of the combined system was arbitrarily chosen to be either a simply supported or a fixed-free beam.

To validate the proposed discretization scheme, the frequency response of combined systems consisting of a simply supported beam and a fixed-free beam carrying various lumped attachments will be considered, in which case $\mathbf{F}$ denotes a vector of forces and torques. To apply Eqs. (2) and (5), matrices $[M_0]$, $[K_0]$, $[U]$, and $[\Lambda]$ are required. Matrices $[M_0]$ and $[K_0]$ are obtained by superimposing the individual element matrices [20] and enforcing the appropriate boundary conditions at the ends. Matrices $[U]$ and $[\Lambda]$ are assembled directly from the exact modes of vibration once the boundary conditions for the beam are specified. For a simply supported beam, its normalized (with respect to ρ) $v_i(x)$, λ_i, and ω_i' are given by

$$v_i(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{i\pi x}{L}\right) \tag{10}$$

$$\lambda_i = (i\pi)^4 \frac{EI}{\rho L^4} \quad \text{and} \quad \omega_i' = (i\pi)^2 \sqrt{\frac{EI}{\rho L^4}} \tag{11}$$

For a fixed-free beam, its normalized $v_i(x)$, λ_i, and ω_i' are given by

$$v_i(x) = \frac{1}{\sqrt{\rho L}} \bigg[\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L} (\sin\beta_i x - \sinh\beta_i x) \bigg] \tag{12}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos\beta_i L \cosh\beta_i L = -1 \tag{13}$$

and

$$\lambda_i = (\beta_i L)^4 \frac{EI}{\rho L^4} \quad \text{and} \quad \omega_i' = (\beta_i L)^2 \sqrt{\frac{EI}{\rho L^4}} \tag{14}$$

The generalized coordinates of a beam element consist of the lateral displacement and the angular rotation (or slope) at the nodes [20]. Hence, if the simply supported or fixed-free beam is discretized into n equal finite elements, there is a total of $N = 2n$ generalized coordinates. Moreover, to assemble $[U]$ of the linear structure, the lateral deflection and the slope at each node must be specified. Fortunately, knowing $v_i(x)$ of the beam, its slope at any point x is

$$\theta_i(x) = \frac{\mathrm{d}}{\mathrm{d}x}[v_i(x)] \tag{15}$$

Once the exact lateral displacements and angular rotations at the nodes have been computed, matrix $[U]$ can be easily assembled, where the elements of the ith column of $[U]$ are obtained by evaluating the ith eigenfunction and its derivative at the appropriate node locations. Finally, the ith element of the diagonal matrix $[\Lambda]$ is given by λ_i.

Consider a combined system subjected to a localized harmonic force input. Our objective is to determine the frequency response of the structure over some specified range of excitation frequencies. Depending on the lower and upper bounds of ω, the mass and stiffness matrices are first updated using appropriately selected exact modes of the bare beam. As a rule of thumb, these exact modes are chosen such that their natural frequencies encompass all the excitation frequencies of interest. For convenience, Table 1 shows the first 20 exact natural frequencies of a uniform simply supported and a fixed-free Euler Bernoulli beam. To minimize the computational cost and time, the fewest number of exact modes is employed to perform the update.

In all of the subsequent examples, the lower and upper bounds of the desired excitation frequencies are specified. In general, all the normal modes are excited when a system is forced. However, if the natural frequencies of the system are well separated, and if the excitation frequency centers around the jth natural frequency of the structure, then it is possible to excite only the jth normal mode and leave all the other normal modes totally unaffected. Physically, this implies that, when the natural frequencies are far apart, and that when the excitation frequency is in the vicinity of a given natural frequency (either slightly below or slightly above the given natural frequency), the response of the structure will be dominated by the jth normal mode. Thus, depending on the linear structure and the lumped attachments, the modification scheme may return a frequency response that remains accurate for excitation frequencies that are slightly beyond the specified bounds. When presenting the frequency responses to illustrate the utility of the proposed discretization scheme, the range of excitation frequencies is purposely enlarged slightly to demonstrate that the validity of the new approach may extend beyond the specified frequency range.

Simply Supported Beam

Consider a simply supported beam carrying a grounded $k_t = 5EI/L$ at $x_a = 0.4L$. The frequency response of the combined system is desired for $\omega < 900\sqrt{EI/(\rho L^4)}$, where the excitation frequencies are less than the 10th natural frequency of a simply supported beam (see Table 1). Thus, the mass and stiffness matrices of the beam are updated using the first 10 exact modes of the simply supported beam. Table 2 shows the first 10 natural frequencies of the combined system, obtained by discretizing the beam into $n = 100$

Table 1 First 20 exact natural frequencies of a uniform simply supported and fixed-free Euler Bernoulli beam; all of the natural frequencies are normalized by dividing by $\sqrt{EI/(\rho L^4)}$

ω_i'	Simply supported	Fixed-free beam
ω_1'	9.86960e + 00	3.51602e + 00
ω_2'	3.94784e + 01	2.20345e + 01
ω_3'	8.88264e + 01	6.16972e + 01
ω_4'	1.57914e + 02	1.20902e + 02
ω_5'	2.46740e + 02	1.99860e + 02
ω_6'	3.55306e + 02	2.98556e + 02
ω_7'	4.83611e + 02	4.16991e + 02
ω_8'	6.31655e + 02	5.55165e + 02
ω_9'	7.99438e + 02	7.13079e + 02
ω_{10}'	9.86960e + 02	8.90732e + 02
ω_{11}'	1.19422e + 03	1.08812e + 03
ω_{12}'	1.42122e + 03	1.30526e + 03
ω_{13}'	1.66796e + 03	1.54213e + 03
ω_{14}'	1.93444e + 03	1.79874e + 03
ω_{15}'	2.22066e + 03	2.07508e + 03
ω_{16}'	2.52662e + 03	2.37117e + 03
ω_{17}'	2.85232e + 03	2.68700e + 03
ω_{18}'	3.19775e + 03	3.02257e + 03
ω_{19}'	3.56293e + 03	3.37787e + 03
ω_{20}'	3.94784e + 03	3.75292e + 03

Table 2 First 10 natural frequencies of a simply supported beam carrying a torsional spring of stiffness $k_t = 5.0EI/L$ at $x_a = 0.4L$

ω_i	FEM, $n = 100$	FEM, $n = 5$	New scheme, $n = 5$, ϵ_i
ω_1	1.02090e + 01	1.02101e + 01 (1.14e-04)	1.02205e + 01 (1.12e-03)
ω_2	4.20719e + 01	4.21509e + 01 (1.88e-03)	4.21697e + 01 (2.32e-03)
ω_3	9.17473e + 01	9.25193e + 01 (8.42e-03)	9.18719e + 01 (1.36e-03)
ω_4	1.58343e + 02	1.61998e + 02 (2.31e-02)	1.58361e + 02 (1.17e-04)
ω_5	2.51344e + 02	2.80163e + 02 (1.15e-01)	2.51549e + 02 (8.15e-04)
ω_6	3.55756e + 02	3.96147e + 02 (1.14e-01)	3.55777e + 02 (5.77e-05)
ω_7	4.86669e + 02	5.82453e + 02 (1.97e-01)	4.86810e + 02 (2.90e-04)
ω_8	6.34796e + 02	8.26840e + 02 (3.03e-01)	6.34949e + 02 (2.41e-04)
ω_9	7.99896e + 02	1.10198e + 03 (3.78e-01)	7.99916e + 02 (2.48e-05)
ω_{10}	9.91767e + 02	1.26579e + 03 (2.76e-01)	9.92037e + 02 (2.71e-04)

Table 3 First 11 natural frequencies of a simply supported beam carrying an oscillator with a rigid body degree of freedom at $x_a = 0.4L$; oscillator parameters are $m = 0.25\rho L$ and $k = 10.0EI/L^3$

ω_i	FEM, $n = 100$	FEM, $n = 5$	New scheme, $n = 5$, ϵ_i
ω_1	5.59053e + 00	5.59055e + 00 (3.27e-06)	5.59060e + 00 (1.18e-05)
ω_2	1.11295e + 01	1.11308e + 01 (1.24e-04)	1.11296e + 01 (4.47e-06)
ω_3	3.95690e + 01	3.96349e + 01 (1.66e-03)	3.95691e + 01 (6.78e-08)
ω_4	8.88656e + 01	8.95719e + 01 (7.94e-03)	8.88656e + 01 (3.08e-08)
ω_5	1.57971e + 02	1.61614e + 02 (2.30e-02)	1.57971e + 02 (1.57e-07)
ω_6	2.46740e + 02	2.73861e + 02 (1.10e-01)	2.46740e + 02 (4.23e-07)
ω_7	3.55331e + 02	3.95338e + 02 (1.12e-01)	3.55331e + 02 (8.74e-07)
ω_8	4.83618e + 02	5.75583e + 02 (1.90e-01)	4.83618e + 02 (1.62e-06)
ω_9	6.31662e + 02	8.17488e + 02 (2.94e-01)	6.31660e + 02 (2.77e-06)
ω_{10}	7.99453e + 02	1.10026e + 03 (3.76e-01)	7.99449e + 02 (4.43e-06)
ω_{11}	9.86960e + 02	1.25499e + 03 (2.71e-01)	9.86960e + 02 (6.75e-06)

and $n = 5$ using the traditional FEM, and into $n = 5$ using the proposed discretization method. For all practical purposes, the natural frequencies obtained with $n = 100$ can be considered "exact." To measure the accuracy of ω_i obtained by using FEM and the new discretization scheme with $n = 5$, a relative error parameter is introduced as follows:

$$\epsilon_i = \left| \frac{\omega_i - (\omega_i)_{100}}{(\omega_i)_{100}} \right| \tag{16}$$

The smaller the ϵ_i, the more accurate the natural frequencies are. The relative errors are shown within the parenthesis. Note that the largest ϵ_i for the new scheme is less than 0.24%, whereas using FEM with $n = 5$, the largest ϵ_i exceeds 37%. Figure 2 shows the frequency response at $0.6L$ along the beam that is subjected to a concentrated harmonic force exerted at $x_f = 0.8L$. Note that for $\omega < 900\sqrt{EI/(\rho L^4)}$, the frequency response given by the new discretization scheme with $n = 5$ (the dashed line) is almost identical to that of FEM with $n = 100$ (the solid line), and all the peaks occur at nearly the same excitation frequencies. These peaks, incidentally, correspond to the natural frequencies of the combined system. For comparison, the frequency response obtained using FEM with $n = 5$ (the dotted line) is also illustrated. Note the large deviation from the exact frequency response for $\omega > 250\sqrt{EI/(\rho L^4)}$.

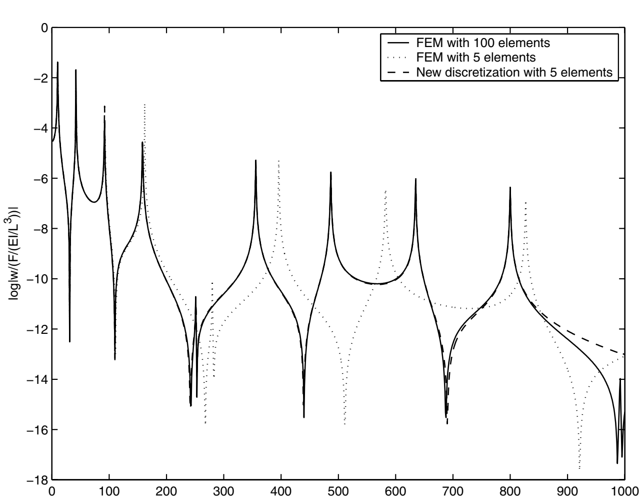

Fig. 2 Frequency response at 0.6L along a simply supported beam carrying a grounded torsional spring.

Consider a simply supported beam carrying an oscillator of parameters $m = 0.25\rho L$ and $k = 10EI/L^3$ with a rigid body degree of freedom attached at $x_a = 0.4L$. The frequency response of the combined system for $\omega < 900\sqrt{EI/(\rho L^4)}$ is desired, and the first 10 exact modes of the bare beam are used to perform the update. Because the oscillator gives rise to an additional degree of freedom, when the beam is discretized into $n = 5$, the combined system consists of a total of 11 natural frequencies. Table 3 shows the first 11 natural frequencies of the combined system obtained by the various methods. Note that the 6th and 11th natural frequencies of the combined system correspond to the 5th and 10th natural frequencies of the simply supported beam, because the attachment location x_a coincides with a node of the 5th and 10th normal mode of the bare beam. In addition, observe that ϵ_i for $i \geq 2$ given by the new discretization scheme are substantially smaller than those of FEM with $n = 5$. Figure 3 illustrates the frequency response at $0.6L$ for $x_f = 0.8L$. Again, the new discretization scheme greatly improves the frequency response of the system, especially for higher excitation frequencies. Comparing the natural frequencies of Table 3 and the peaks in the frequency response of Fig. 3, note that the 6th and 11th natural frequencies of the combined system (or the 5th and 10th natural frequencies of the simply supported beam) do not appear as peaks in the frequency response curve. In general, when a beam is excited at the ith natural frequency, its response is dominated by the ith mode shape. However, if x_f coincides with a node for a particular mode shape, the effect of that mode on the response of the beam will be eliminated. Because $x_a = 0.4L$ and $x_f = 0.8L$, which correspond to the nodes of the 5th and 10th mode shapes of the simply supported beam, the frequency response of the combined system will not exhibit peaks at $246.740\sqrt{EI/(\rho L^4)}$ and $986.960\sqrt{EI/(\rho L^4)}$.

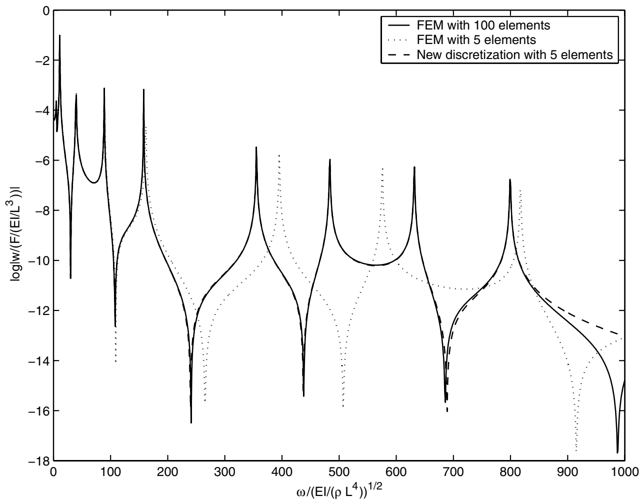

Fig. 3 Frequency response at 0.6L along a simply supported beam carrying an oscillator with a rigid body degree of freedom.

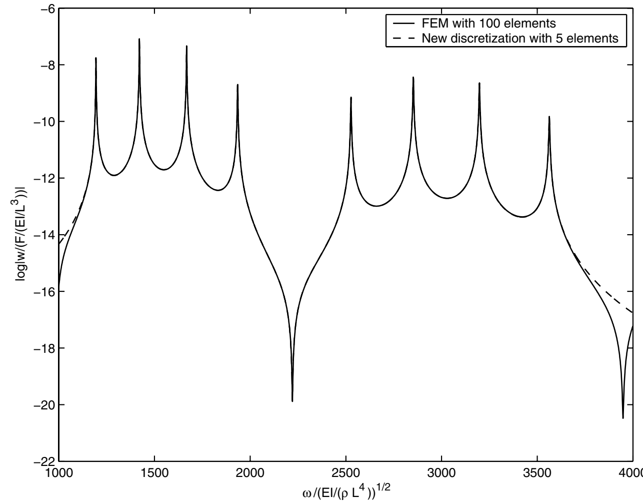

Fig. 4 Frequency response at 0.2L along a simply supported beam carrying a grounded translational spring.

From the previous examples, one notes that, by updating the mass and stiffness matrices with the first 10 exact modes of a simply supported beam, the new discretization scheme returns natural frequencies for the combined system that are nearly identical to those obtained with $n = 100$, even though the beam is discretized into $n = 5$ only. Moreover, the corresponding frequency response of the combined system at any point along the structure is greatly improved when the excitation frequencies are less than the 10th natural frequency of the simply supported beam. The proposed scheme is easy to apply, and can be used to update the mass and stiffness matrices of the bare beam using any set of selected exact modes of vibration. Thus, if the frequency response of the combined system at higher excitation frequencies is desired, instead of discretizing the beam into many finite elements, one can modify the mass and stiffness matrices using only a selected set of the higher exact modes. These modes are chosen so that their natural frequencies encompass the entire range of excitation frequencies of interest.

Consider a simply supported beam carrying a grounded $k = 120EI/L^3$ at $x_a = 0.4L$. For a given application, the frequency response of the combined system for $1200\sqrt{EI/(\rho L^4)} < \omega < 3700\sqrt{EI/(\rho L^4)}$ is desired. Thus, the mass and stiffness matrices are updated using the 11th–20th exact modes of a bare simply supported beam. Table 4 shows the 11th–20th natural frequencies of the combined system given by FEM with $n = 100$ and the new discretization scheme with $n = 5$. Note that the proposed method yields natural frequencies that are nearly identical to those of FEM with $n = 100$. For brevity, tables of natural frequencies will not be given for any of the subsequent examples. In all cases, the same trend is observed, namely that the proposed discretization scheme yields natural frequencies that are nearly identical to those obtained by dividing the beam into $n = 100$. Figure 4 shows the frequency response at $0.2L$ along the beam, where $x_f = 0.8L$. Note that, for

Table 4 Eleventh to the 20th natural frequencies of a simply supported beam carrying a grounded spring of stiffness $k = 120EI/L^3$ at $x_a = 0.4L$

ω_i	FEM, $n = 100$	New scheme, $n = 5$, ϵ_i
ω_{11}	1.19432e + 03	1.19431e + 03 (9.95e-06)
ω_{12}	1.42127e + 03	1.42125e + 03 (1.40e-05)
ω_{13}	1.66802e + 03	1.66799e + 03 (1.92e-05)
ω_{14}	1.93455e + 03	1.93450e + 03 (2.59e-05)
ω_{15}	2.22073e + 03	2.22066e + 03 (3.40e-05)
ω_{16}	2.52677e + 03	2.52666e + 03 (4.40e-05)
ω_{17}	2.85249e + 03	2.85233e + 03 (5.60e-05)
ω_{18}	3.19799e + 03	3.19776e + 03 (7.04e-05)
ω_{19}	3.56326e + 03	3.56296e + 03 (8.73e-05)
ω_{20}	3.94826e + 03	3.94784e + 03 (1.07e-04)

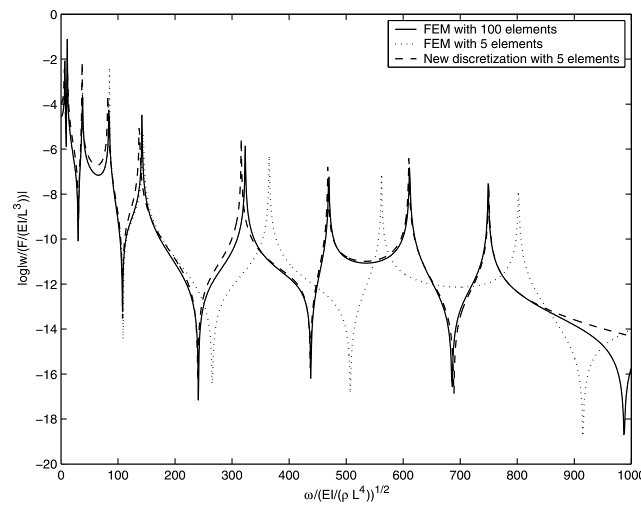

Fig. 5 Frequency response at 0.6L along a simply supported beam carrying an oscillator (m_1 and k_1) with a rigid body degree of freedom at $x_{a1} = 0.4L$, and an oscillator (m_2 and k_2) with no rigid body degree of freedom at $x_{a2} = 0.6L$.

excitation frequencies that lie between the specified bounds, the frequency response given by the new discretization scheme with $n = 5$ (the dashed line) is indistinguishable from that of FEM with $n = 100$ (the solid line), and the peaks occur at almost exactly the same locations.

To further exhibit the utility of the proposed method, cases in which the beam carries multiple attachments are provided. Consider now a simply supported beam carrying an oscillator ($m_1 = 0.15\rho L$ and $k_1 = 12EI/L^3$) with a rigid body degree of freedom and an oscillator ($m_2 = 0.22\rho L$ and $k_2 = 8EI/L^3$) with no rigid body degree of freedom at $x_{a1} = 0.4L$ and $x_{a2} = 0.6L$, respectively. Assuming the frequency response of the combined system for $\omega < 900\sqrt{EI/(\rho L^4)}$ is desired, the mass and stiffness matrices are updated using the first 10 exact modes of the bare beam. Figure 5 shows the frequency response of the beam at $0.6L$, where $x_f = 0.8L$. The new discretization scheme with $n = 5$ again returns a frequency response that tracks that of FEM with $n = 100$, and the peaks occur at nearly the same locations. The frequency response obtained by using FEM with $n = 5$, on the other hand, fails to track that obtained by using FEM with $n = 100$ for $\omega > 200\sqrt{EI/(\rho L^4)}$.

As a second example of multiple-attachment cases, consider a simply supported beam carrying two attachments, a grounded $k_1 = 20EI/L^3$ at $x_{a1} = 0.4L$, and an oscillator of parameters

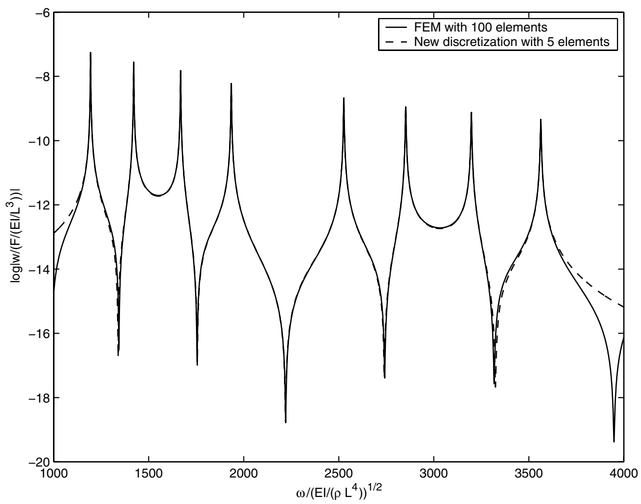

Fig. 6 Frequency response at $0.6L$ along a simply supported beam carrying a grounded spring k_1 at $x_{a1} = 0.4L$, and an oscillator (m_2 and k_2) with a rigid body degree of freedom at $x_{a2} = 0.6L$.

$m_2 = 0.25\rho L$ and $k_2 = 50EI/L^3$ with a rigid body degree of freedom at $x_{a2} = 0.6L$. The frequency response of the combined system is desired, where $1200\sqrt{EI/(\rho L^4)} < \omega < 3700\sqrt{EI/(\rho L^4)}$. Thus, the mass and stiffness matrices are updated using the 11th–20th exact modes of a bare beam. Figure 6 shows the frequency response at $0.6L$ for $x_f = 0.8L$. Note that the peaks occur at nearly the identical locations, and that the frequency response obtained by applying the new discretization scheme with $n = 5$ tracks that obtained by dividing the beam into $n = 100$ almost exactly over the range of specified frequencies. The results again demonstrate that with only $n = 5$, the proposed discretization scheme yields accurate natural frequencies and frequency response.

Fixed-Free Beam

Consider now a fixed-free beam carrying a grounded $k_t = 5EI/L$ at $x_a = 0.4L$. The excitation frequencies of interest are less than $850\sqrt{EI/(\rho L^4)}$, which is less than the 10th natural frequency of a fixed-free beam (see Table 1). Thus, the mass and stiffness matrices are updated using the first 10 exact modes of a fixed-free beam. Figure 7 shows the frequency response at $0.4L$ for $x_f = 0.8L$. Interestingly, the updated model yields a frequency response that is accurate for excitation frequencies up to $1000\sqrt{EI/(\rho L^4)}$, greater than the 10th natural frequency of a fixed-free beam. Note that the peaks and the frequency responses given by FEM with $n = 100$ and the new discretization scheme with $n = 5$ almost coincide. The frequency response given by FEM with $n = 5$, on the other hand, is only accurate for $\omega < 300\sqrt{EI/(\rho L^4)}$.

Consider now a fixed-free beam carrying an oscillator of parameters $m = 0.25\rho L$ and $k = 10EI/L^3$ with a rigid body degree of freedom at $x_a = 0.4L$. The frequency response for $\omega < 850\sqrt{EI/(\rho L^4)}$ is again desired. Figure 8 shows the frequency response at $0.4L$ for $x_f = 0.8L$. The new discretization scheme with $n = 5$ again returns a frequency response that is nearly identical to that of FEM with $n = 100$. The frequency response obtained by using FEM with $n = 5$, however, fails to track the FEM solution with 100 elements for $\omega > 300\sqrt{EI/(\rho L^4)}$.

Consider a combined system consisting of a fixed-free beam carrying a grounded $k = 120EI/L^3$ at $x_a = 0.4L$. The frequency response for excitation frequencies that lie between $1250\sqrt{EI/(\rho L^4)}$ and $3700\sqrt{EI/(\rho L^4)}$ is required. Thus, the 11th–20th exact modes of a fixed-free beam are used to perform the updates. Figure 9 shows the frequency responses at $0.4L$ for $x_f = 0.8L$. Note that, using the new discretization scheme, the frequency response is accurate only for $\omega < 2100\sqrt{EI/(\rho L^4)}$. Figure 9 appears to show that the discretization scheme fails for a

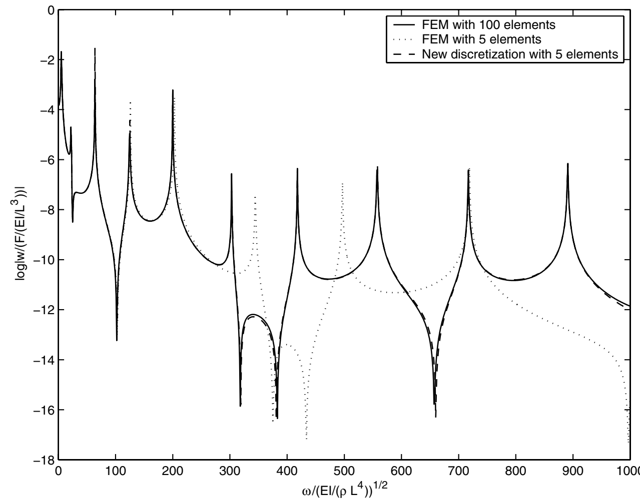

Fig. 7 Frequency response at $0.4L$ along a fixed-free beam carrying a grounded torsional spring.

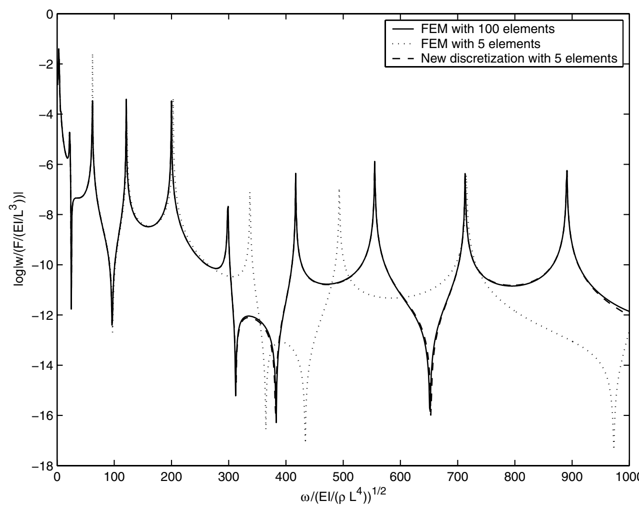

Fig. 8 Frequency response at $0.4L$ along a fixed-free beam carrying an oscillator with a rigid body degree of freedom.

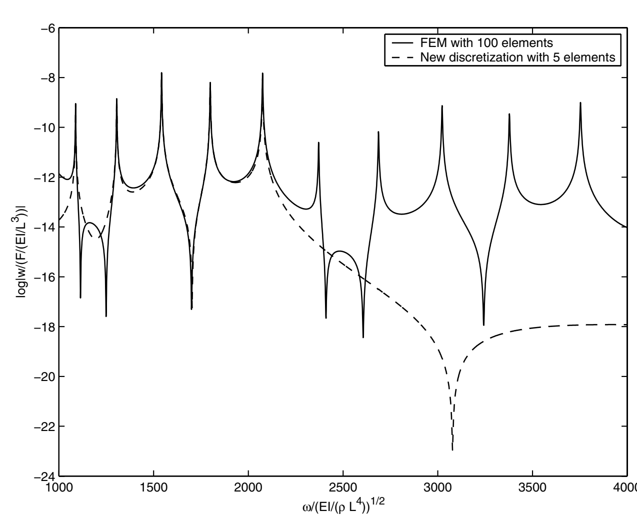

Fig. 9 Frequency response at $0.4L$ along a fixed-free beam carrying a grounded spring. The mass and stiffness matrices are updated by using the 11th–20th exact modes [see Eq. (12)] of a fixed-free beam.

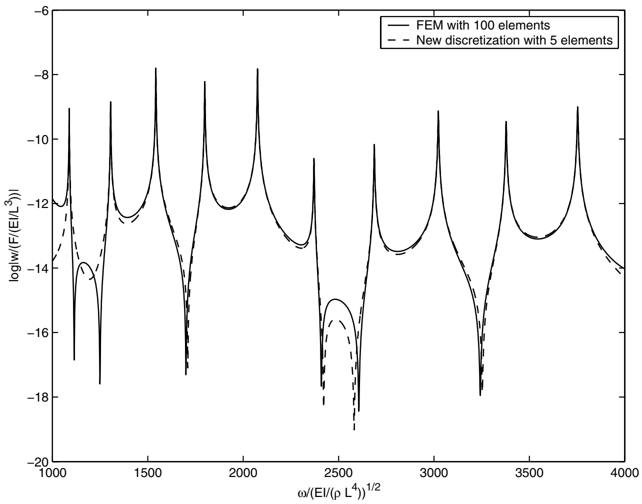

Fig. 10 Frequency response at $0.4L$ along a fixed-free beam carrying a grounded spring. The mass and stiffness matrices are updated by using the 11th–20th exact modes [see Eq. (17)] of a fixed-free beam.

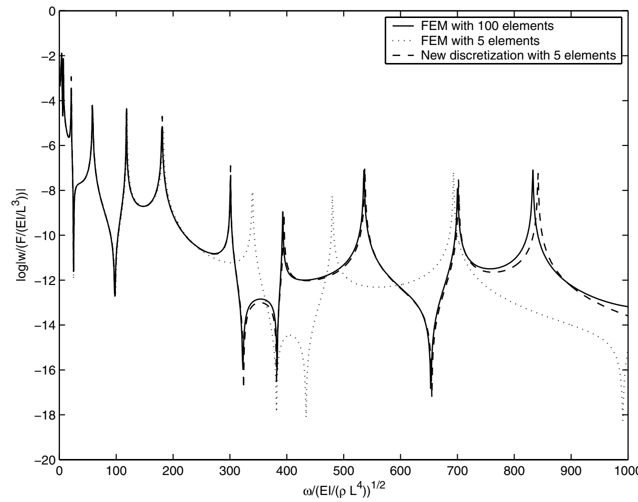

Fig. 11 Frequency response at $0.4L$ along a fixed-free beam carrying a grounded torsional spring k_t at $x_{a1} = 0.2L$, a lumped mass m at $x_{a2} = 0.4L$, and an oscillator (m_1 and k_1) with a rigid body degree of freedom at $x_{a3} = 0.6L$

fixed-free beam when the excitation frequency exceeds a specific value. The failure of the proposed method for the case under consideration, however, can be explained by noting that Eq. (12), which corresponds to the exact eigenfunctions of a fixed-free beam, consists of hyperbolic functions, and it becomes numerically unstable when $i > 14$.

To circumvent this numerical difficulty, an alternative expression for the exact normalized eigenfunctions of a fixed-free beam that remains numerically stable at the higher modes will be used, and it is given by [21]

$$v_i(x) = \frac{1}{\sqrt{\rho L}}[a_i(x) - b_i(x) + (1 + \mu_i)c_i(x) - \mu_i d_i(x)] \tag{17}$$

where

$$a_i(x) = e^{-\beta_i x}, \qquad b_i(x) = \cos\beta_i x \qquad c_i(x) = \sin\beta_i x,$$
$$d_i(x) = \sinh\beta_i x \qquad \mu_i = \frac{-(e^{-\beta_i L} + \cos\beta_i L + \sin\beta_i L)}{\cosh\beta_i L + \cos\beta_i L} \tag{18}$$

and $\beta_i L$ satisfies Eq. (13). To avoid numerical ill-conditioning, the finite element mass and stiffness matrices of a fixed-free beam are updated using the 11th–20th eigenfunctions of Eq. (17). Figure 10 shows the frequency response of the combined system at $0.4L$. Note the drastic improvement in the frequency response compared with that shown in Fig. 9, where Eq. (12) was used to perform the update. Moreover, for the specified range of excitation frequencies, the frequency response obtained by using the modified finite element method with $n = 5$ tracks the frequency response obtained by the traditional FEM with $n = 100$ accurately, proving the computationally efficiency of the proposed discretization scheme.

As the last example, consider a fixed-free beam carrying a grounded $k_t = 5EI/L$ at $x_{a1} = 0.2L$, $m = 0.2\rho L$ at $x_{a2} = 0.4L$, and an oscillator of parameters $m_1 = 0.25\rho L$ and $k_1 = 10EI/L^3$ with a rigid body degree of freedom at $x_{a3} = 0.6L$. The excitation frequencies of interest are $\omega < 850\sqrt{EI/(\rho L^4)}$, and the updates are performed using the first 10 exact modes of a fixed-free beam. Figure 11 shows the frequency response at $0.4L$ for $x_f = 0.8L$. Note that the frequency response obtained by applying the new scheme with $n = 5$ tracks the FEM result with $n = 100$ well over the entire range of excitation frequencies considered. Moreover, note that their peak locations are nearly identical, except for the last one. However, even for this worst case, the natural frequencies obtained using FEM with $n = 100$ and the new discretization scheme with $n = 5$ are $833.316\sqrt{EI/(\rho L^4)}$ and $841.787\sqrt{EI/(\rho L^4)}$, a relative error of less than 1.02%. For comparison, the frequency response obtained using FEM with $n = 5$ is also illustrated. It tracks the FEM frequency response curve with $n = 100$ elements only for $\omega < 280\sqrt{EI/(\rho L^4)}$. Figure 11 again clearly validates the utility of the newly modified finite element discretization scheme.

Finally, for a complicated real structure where there is no analytical formulation for the mode shapes or natural frequencies, one can use the modes of vibration of its FE model with a fine mesh to update its FE model with a coarse mesh. Of course, the normal modes and the corresponding natural frequencies that are used to perform the update must be carefully chosen. Specifically, the natural frequencies must encompass the excitation frequencies of interest. Thus, instead of using a fine mesh to compute the frequency response of a combined system at high excitation frequencies, the modified finite element method allows one to accurately determine the frequency response of the combined system using only a few finite elements. Once an updated coarse FE model has been constructed, it can be used predictively for untested arbitrary loading conditions. This will be pursued in a future paper.

Conclusions

A modified finite element discretization scheme is proposed that allows one to accurately determine the frequency response of a combined system, which consists of a linear structure carrying various lumped attachments, while using a coarse mesh. Knowing the range of excitation frequencies of interest, the finite element model of the linear structure is first updated using the exact modes whose natural frequencies encompass the excitation frequencies. The mass and stiffness updating schemes minimize changes in the finite element system matrices such that the finite element eigendata coincide with the exact modes of vibration. After performing the updates, the traditional finite element assembling technique is exploited to include the lumped attachments to form the global mass and stiffness matrices, after which the frequency response of the combined system at any point along the linear structure can be easily computed. Numerical experiments show that the proposed discretization scheme yields very accurate frequency response over any range of excitation frequencies while using only a limited number of finite elements.

References

[1] Pomazal, R. J., and Snyder, V. W., "Local Modifications of Damped Linear Systems," *AIAA Journal*, Vol. 9, No. 11, 1971, pp. 2216–2221. doi:10.2514/3.50028

[2] Goel, R. P., "Free Vibrations of a Beam-Mass System with Elastically Restrained Ends," *Journal of Sound and Vibration*, Vol. 47, No. 1, 1976, pp. 9–14. doi:10.1016/0022-460X(76)90404-1

[3] Dowell, E. H., "On Some General Properties of Combined Dynamic Systems," *Journal of Applied Mechanics*, Vol. 46, No. 1, 1979, pp. 206–209.

[4] Nicholson, J. W., and Bergman, L. A., "Free Vibration of Combined Dynamic System," *Journal of Engineering Mechanics*, Vol. 112, No. 1, 1986, pp. 1–13.

[5] Ercoli, L., and Laura, P. A. A., "Analytical and Experimental Investigation on Continuous Beams Carrying Elastically Mounted Masses," *Journal of Sound and Vibration*, Vol. 114, No. 3, 1987, pp. 519–533.
doi:10.1016/S0022-460X(87)80021-4

[6] Wu, J. S., and Lin, T. L., "Free Vibration Analysis of a Uniform Cantilever Beam with Point Masses by an Analytical-and-Numerical-Combined Method," *Journal of Sound and Vibration*, Vol. 136, No. 2, 1990, pp. 201–213.
doi:10.1016/0022-460X(90)90851-P

[7] Rossi, R. E., Laura, P. A. A., Avalos, D. R., and Larrondo, H., "Free Vibrations of Timoshenko Beams Carrying Elastically Mounted, Concentrated Masses," *Journal of Sound and Vibration*, Vol. 165, No. 2, 1993, pp. 209–223.
doi:10.1006/jsvi.1993.1254

[8] Kukla, S., and Posiadała, B., "Free Vibrations of Beams with Elastically Mounted Masses," *Journal of Sound and Vibration*, Vol. 175, No. 4, 1994, pp. 557–564.
doi:10.1006/jsvi.1994.1345

[9] Lueschen, G. G. G., Bergman, L. A., and McFarland, D. M., "Green's Functions for Uniform Timoshenko Beams," *Journal of Sound and Vibration*, Vol. 194, No. 1, 1996, pp. 93–102.
doi:10.1006/jsvi.1996.0346

[10] Posiadała, B., "Free Vibrations of Uniform Timoshenko Beams with Attachments," *Journal of Sound and Vibration*, Vol. 204, No. 2, 1997, pp. 359–369.
doi:10.1006/jsvi.1997.0952

[11] Kukla, S., "Application of Green Functions in Frequency Analysis of Timoshenko Beams with Oscillators," *Journal of Sound and Vibration*, Vol. 205, No. 3, 1997, pp. 355–363.
doi:10.1006/jsvi.1997.0971

[12] Gürgöze, M., "On the Alternative Formulations of the Frequency Equation of a Bernoulli-Euler Beam to Which Several Spring-Mass Systems are Attached In-Span," *Journal of Sound and Vibration*, Vol. 217, No. 3, 1998, pp. 585–595.
doi:10.1006/jsvi.1998.1796

[13] Cha, P. D., and Wong, W. C., "A Novel Approach to Determine the Frequency Equations of Combined Dynamic Systems," *Journal of Sound and Vibration*, Vol. 219, No. 4, 1999, pp. 689–706.
doi:10.1006/jsvi.1998.1916

[14] Wu, J. S., and Chou, H. M., "A New Approach for Determining the Natural Frequencies and Mode Shapes of a Uniform Beam Carrying any Number of Sprung Masses," *Journal of Sound and Vibration*, Vol. 220, No. 3, 1999, pp. 451–468.
doi:10.1006/jsvi.1998.1958

[15] Chang, T. P., Chang, F. I., and Liu, M. F., "On the Eigenvalues of a Viscously Damped Simple Beam Carrying Point Masses and Springs," *Journal of Sound and Vibration*, Vol. 240, No. 4, 2001, pp. 769–778.
doi:10.1006/jsvi.2000.3186

[16] Cha, P. D., "Eigenvalues of a Linear Elastica Carrying Lumped Masses, Springs and Viscous Dampers," *Journal of Sound and Vibration*, Vol. 257, No. 4, 2002, pp. 798–808.
doi:10.1006/jsvi.2001.4242

[17] Berman, A., and Nagy, E. J., "Improvement of a Large Analytical Model Using Test Data," *AIAA Journal*, Vol. 21, No. 8, 1983, pp. 1168–1173.
doi:10.2514/3.60140

[18] Berman, A., "Mass Matrix Correction Using an Incomplete Set of Measured Modes," *AIAA Journal*, Vol. 17, No. 10, 1979, pp. 1147–1148.
doi:10.2514/3.61290

[19] Wei, F. S., "Stiffness Matrix Correction from Incomplete Test Data," *AIAA Journal*, Vol. 18, No. 10, 1980, pp. 1274–1275.
doi:10.2514/3.7724

[20] Meirovitch, L., *Fundamentals of Vibrations*, McGraw–Hill, New York, 2001, pp. 574–578.

[21] Gonçalves, P. J. P., Brennan, M. J., and Elliott, S. J., "Numerical Evaluation of High-Order Modes of Vibration in Uniform Euler-Bernoulli Beams," *Journal of Sound and Vibration*, Vol. 301, Nos. 3–5, 2007, pp. 1035–1039.
doi:10.1016/j.jsv.2006.10.012

J. Cooper
Associate Editor